Further reflections on candidate “rule-them-all” equations [1]

Domenico Cantone, Eugenio G. Omodeo

Università degli Studi di Trieste

Dip. Matematica e Geoscienze — DMI

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MEMORANDUM
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ONE EQUATION TO RULE THEM ALL

Martin Davis

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DIVISION OF MATHEMATICS

ONE EQUATION TO RULE THEM ALL*

Martin Davis

Mathematics Department, New York University, New York, N.Y.

Let H stand for the assertion:
The equation

$$9\left(u^2 + 7v^2\right)^2 - 7\left(r^2 + 7s^2\right)^2 = 2 \tag{1}$$

has no solution in non-negative integers except the trivial $u = r = 1$, $v = s = 0$.

We must leave the truth of H open; however, in this note we shall prove:

H implies that there is no uniform algorithm for testing polynomial Diophantine equations for solvability in positive integers, i.e., that Hilbert's tenth problem is unsolvable.†

As will be seen our methods yield a considerably stronger result. Also, the reader will see how the methods used can be readily adapted to obtain various other hypotheses about which the same can be demonstrated as we do here for H. We will conclude with a report on numerical calculations (some using the JOSS system) seeking counterexamples to H.

We wish to acknowledge helpful discussions with Robert DiPaola, Oliver Gross, Hilary Putnam, Norman Shapiro, and Joel Spencer. The ideas of Section 3 come from unpublished joint work done with Hilary Putnam during the summer of 1960.‡

1. SOME PROPERTIES OF SOLUTIONS OF THE PELL EQUATION $x^2 - 7y^2 = 1$

Below, p always is a prime number.

Lemma 1. The successive non-negative integer solutions of $x^2 - 7y^2 = 1$ are given (for $n \geq 0$) by

$$x_n + y_n\sqrt{7} = (8 + 3\sqrt{7})^n$$

Proof. By Theorem 104 of Reference 8, we have:

*This paper was presented at a meeting of the Division on March 7, 1968.
†The research in this paper was partly sponsored by the Air Force Office of Scientific Research and was partly conducted while the author was a consultant for the RAND Corporation. "JOSS" is the RAND Corporation's trademark for its computing system. For previous results and general background, see Chapter 7 of Reference 1, and References 2-9.
‡The same material is referred to as "Proposition 2" in Reference 3. As stated, the "proposition" requires the correction: Replace "$r^2 + ds^2$" by "$a_1 a_1'(r^2 + ds^2)$."

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Outline

1. *Existentially definable*, in particular *Diophantine*, sets
   - single-fold and finite-fold existential representations
2. Relations of *exponential growth*
3. Quaternary quartic "rule-them-all" equations. E.g., over $\mathbb{N}$:
   $3\ \left(r^2+3\,s^2\right)^2-\left(u^2+3\,v^2\right)^2\ =\ 2$
4. - Integers expressible in a specific quadratic form, e.g. $u^2+3\,v^2$
   - A novel quaternary quartic (over $\mathbb{Z}$):
     $11\ \left(s^2+s\,r+3\,r^2\right)^2-\left(v^2+v\,u+3\,u^2\right)^2\ =\ 2$
5. Is the exponential-growth relation

$$\left\{\langle u, \boldsymbol{y}_{2^{2\,\ell+1}}(11)\rangle\ :\ \ell\geqslant 2\ \&\ u\geqslant 2^{2\,\ell+2}\right\}$$

Diophantine?

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Diophantine? $\boxed{11\,y^2+1=\square}$

The finite-fold-ness issue

A relation $\mathcal{R} \subseteq \mathbb{N}^n$ is said to be ***existentially definable in terms of some relation*** $\mathcal{J}(\bullet, \ldots, \bullet)$ iff

$$\mathcal{R}(a_1, \ldots, a_n) \iff$$

$$(\exists x_1 \cdots \exists x_m)\, \varphi(\, a_1, \ldots, a_n \,,\, x_1, \ldots, x_m \,)$$

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When $\mathcal{J}$ is absent, $\mathcal{R}$ is also called ***Diophantine***.

Single-fold existential definitions

An existential definition

$$\exists \vec{x} \quad \varphi(\, \vec{a},\, \vec{x}\,)$$

(as above) is said to be ***single-fold*** if

$$\forall \vec{a}\, \forall \vec{x}\, \forall \vec{y} \Big[\quad \varphi(\, \vec{a},\, \vec{x}\,) \ \&\ \varphi(\, \vec{a},\, \vec{y}) \implies \vec{x} = \vec{y} \quad \Big]$$

(i.e., $\varphi(\, a_1, \ldots, a_n\,,\, x_1, \ldots, x_m\,)$ never has multiple solutions).

Singlefold-ness

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Finite-fold existential definitions

The definition of ***finite-fold***-ness is akin:

To each $\vec{a}$ there must correspond a *finite* number of solutions.

Now consider listable[2] (aka effectively enumerable) sets.

[2] **Def.** A set is *listable* if its elements can be generated exhaustively by an algorithmic (perhaps non-terminating) procedure.

Two important theorems
(Davis, Putnam, Robinson, Matiyasevich)

DPR theorem (See [DPR61])

Each listable set is *existentially definable in terms of exponentiation* (which is the rel. consisting of all triples $\langle b, n, c\rangle$ such that $b^n = c$).

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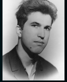
Significant improvement to DPR (see [Mat74], [JM84])

Each listable set admits an existential *single-fold* repr'tion in terms of exponentiation.

(See also [Dav93])

[2] **Def.** A set is *listable* if its elements can be generated exhaustively by an algorithmic (perhaps non-terminating) procedure.

Open p.: Does exponentiation admit a single-fold (or at least finite-fold) Diophantine definition ?



"After the DPR-theorem was proved in 1961, in order to establish the existence of Diophantine representations for *every* effectively enumerable set it was sufficient to find a Diophantine representation for *one particular* set of triples

$$\{ \langle a, b, c \rangle \quad | \quad a = b^c \} \ . \tag{12}$$

Today we are in a similar position with respect to single-fold (and finite-fold) Diophantine representations: now that we can construct single-fold exponential Diophantine representations for all effectively enumerable sets, in order to transform them into single-fold (or finite-fold) genuinely Diophantine representations, it would be sufficient to find a single-fold (or, respectively, finite-fold) Diophantine representation for the same set of triples (12) $\cdots$"

[Mat10, p. 748]

Relations of exponential growth

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Dyadic exponential-growth relations

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1. $\mathcal{J}(u,v) \implies v < u^u$;
2. $\forall\, \ell\, (\exists\, u\,,\; v)\big[\, \mathcal{J}(\,u\,,\; v\,)\;\&\; u^{\ell} < v\,\big]$.

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Historical example — ⌣ Diophantine! ⌣

Take

$$\mathcal{J} = \big\{ \langle u, F_{2u} \rangle \mid u > 1 \big\} ,$$

where

$$F_0 = 0 , \quad F_1 = 1 , \quad F_{\ell+2} = F_{\ell+1} + F_\ell ,$$

for $\ell = 0, 1, 2, \ldots$ (See [Mat70b])

Diophantine reduction of exponentiation to <u>any</u> $\mathcal{J}$ of exponential growth [Rob52]

$$
\begin{aligned}
& b^n = c \iff (\exists\, w,\, h,\, a,\, d,\, \ell,\, u,\, v,\, s,\, q,\, z)\Big[\\
& (c-1)^2 + b + n = 0 \;\vee\; c + b + (n - z - 1)^2 = 0 \;\vee \\
& \Big(b \geqslant 1 \;\&\; c = z + 1 \\
& \quad \&\; w > b \;\&\; w > n \;\&\; \boxed{Q(w,\, h) = q^2} \;\&\; a \geqslant h \;\&\; a > c \\
& \quad \&\; u^2 = (a^2\, b^2 - 1)\, v^2 + 1 \;\&\; \ell \leqslant d \;\&\; \lfloor c = u/\ell \rfloor \\
& \quad \&\; \ell^2 = (a^2 - 1)\,\big(n + (a-1)\, s\big)^2 + 1 \;\&\; \boxed{\boxed{\mathcal{J}(a, d)}} \Big)\Big]
\end{aligned}
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Here:

1. $Q(x, y) = \square \implies y > x^x$;
2. $\forall x\, \exists y \quad Q(x, y) = \square$;

(See [MR75])

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2. $\forall x\, \exists y \quad Q(x, y) = \Box$;

e.g., $Q \leftrightharpoons (x+2)^3\,(x+4)\,(y+1)^2 + 1$. (See [MR75])

Davis's quaternary quartic equation and its siblings

How many solutions has Davis's eq. $9\left(u^2+7v^2\right)^2-7\left(r^2+7s^2\right)^2 = 2$?

[Dav68] takes into account the sequence

$$\langle y_i \rangle_{i\in\mathbb{N}} = \langle 0, 3,\ 48,\ 640, \dots \rangle,$$

such that y_i is the $(i+1)$-st solution of the Pell equation

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Through the study of integers expressible in the form $u^2+7\,v^2$, Davis proves that the exponential-growth relation

$$\left\{ \langle u\,,\ y_{2^\ell} \rangle\ :\ \ell \geqslant 0\ \&\ u \geqslant 2^\ell\ \&\ u > 16 \right\}$$

is *Diophantine* $\boxed{\textbf{if}}$ the equation

$$9\,(u^2+7\,v^2)^2-7\,(r^2+7\,s^2)^2 = 2$$

has no solution in $\mathbb{N}$ except the trivial one: $u=r=1\,,\ v=s=0\,.$

$$9\ \left(u^2+7\,v^2\right)^2-7\ \left(r^2+7\,s^2\right)^2\ =\ 2\ \ ?$$

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$$\boldsymbol{y}(7)\quad=\quad\langle y_i\rangle_{i\in\mathbb{N}}\quad=\quad\langle 0,3,\ 48,\ 640,\ldots\rangle\,,$$

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$$\boxed{\boldsymbol{y}_{2^\ell}(7)}$$

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How many solutions has Davis's eq.

$$9\ \left(u^2+7\,v^2\right)^2-7\ \left(r^2+7\,s^2\right)^2\ =\ 2\ \text{?}$$

Nontrivial solutions exist [SW95]... ☹ ☺ ... Even so:

From [DMR76]: "It can be shown that a *singlefold* Diophantine representation of $a = 2^c$ can be constructed if the equation

$$9\ \left(u^2+7\,v^2\right)^2-7\ \left(r^2+7\,s^2\right)^2\ =\ 2$$

has only a *finite number* of solutions."

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$$\boxed{\textit{`rule-them-all' equation}}$$

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Today there are four (potentially eight) competitors:

- -2 $\quad 2\,(r^2+2\,s^2)^2-(u^2+2\,v^2)^2 \;=\; 1$
- -3 $\quad 3\,(r^2+3\,s^2)^2-(u^2+3\,v^2)^2 \;=\; 2$
- -7 $\quad 9\,(u^2+7\,v^2)^2-7\,(r^2+7\,s^2)^2 \;=\; 2$
- -11 $\quad 11\,(r^2+r\,s+3\,s^2)^2-(v^2+v\,u+3\,u^2)^2 \;=\; 2$ (over $\mathbb{Z}$)

Each of these stems from a square-free rational integer $d>1$ such that the integers of $\mathbb{Q}(\sqrt{-d})$ form a unique-factorization domain.

Solutions to our candidate rule-them-all eq's

Trivial solutions

A solution in $\mathbb{N}$, for each of the four candidates, is:

$$r = u = 1\,, \qquad s = v = 0\,.$$

When the discriminant is -11, the trivial solutions in $\mathbb{Z}$ are:
$s = 0$, $r \in \{-1, 1\}$ and either $v = 0$, $u \in \{-1, 1\}$ or $u = 1$, $v = -1$.

Non-trivial solutions (in $\mathbb{N}$)

At least 50 solutions were found for the discriminant -7.
Two non-trivial solutions for the discriminant -3 were detected, and kindly communicated to us, by Boris Z. Moroz (Rheinische Friedrich-Wilhelms-Universität Bonn) and Carsten Roschinski:

$$\begin{array}{cccc} r = 16\,, & s = 25\,, & u = 4\,, & v = 35\,; \\ r = 124088\,, & s = 7307\,, & u = 134788\,, & v = 54097\,. \end{array}$$

Integers representable in the form $\upsilon^2 + \upsilon\, u + 3\, u^2$

To each of our discriminants

$$-2\,,\quad -3\,,\quad -7\,,\quad -11\,,$$

there corresponds a notion of *representable number*; to wit, a positive integer which can be written in the respective quadratic form:

- -2 $u^2 + 2\,v^2$
- -3 $u^2 + 3\,v^2$
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- -11 $\boxed{v^2 + v\,u + 3\,u^2}$

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with $u\,,\,v \in \mathbb{Z}$.

Clue: Things are so because the integers of an imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ form the ring:

$$\begin{cases} \mathbb{Z}\left[\,\sqrt{-d}\,\right] & \text{if} \quad d \equiv 1, 2 \pmod{4}\,, \\ \mathbb{Z}\left[\,(1 + \sqrt{-d})\,/\,2\,\right] & \text{if} \quad d \equiv 3 \pmod{4}\,. \end{cases}$$

Call a prime number p *poison* if one of the following congruences holds:

$$p \equiv 2\,,\ 6\,,\ 7\,,\ 8\,,\ 10 \pmod{11}\,.$$

Primes of the form $\upsilon^2 + \upsilon\, u + 3\, u^2$

Every *non*-poison prime is so representable

Positive integers of the form $\upsilon^2 + \upsilon\, u + 3\, u^2$

A positive integer x is so representable if and only if there is *no* poison prime dividing it to an odd power

Now consider the increasing sequence $\langle y_i \rangle_{i \in \mathbb{N}} = \langle 0, 3, 60, 1197, \ldots \rangle$ of all solutions to the equation

$$11\, y^2 + 1 \quad = \quad \square \; .$$

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Then (denoting by $\langle x_i \rangle_{i \in \mathbb{N}}$ the associated seq. $\langle 1, 10, 199, 3970, \ldots \rangle$):

- if $y_{2^\ell\,(2\,h+1)}$ is representable, so are $\overbrace{3\, x_h + 11\, y_h \text{ and } x_h + 3\, y_h}^{\textit{coprime}}$

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- if y_n is representable for some $n > 0$ *not* a power of 2, then the system
$$\begin{cases} X^2 - 11\, Y^2 & = & 1\,, \\ 3\, X + 11\, Y & = & \upsilon^2 + \upsilon\, u + 3\, u^2\,, \\ X + \;\; 3\, Y & = & r^2 + r\, s + 3\, s^2 \end{cases}$$
has a solution in $\mathbb{N}$ for which $Y \neq 0$

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  has a solution in $\mathbb{N}$ for which $Y \neq 0$; $\therefore$, the eq.

$$11\, \left(r^2 + r\, s + 3\, s^2\right)^2 - \left(\upsilon^2 + \upsilon\, u + 3\, u^2\right)^2 \quad = \quad 2$$

  has a solution with either

$$r^2 + r\, s + 3\, s^2 \neq 1 \ \text{ or } \upsilon^2 + \upsilon\, u + 3\, u^2 \neq 3\,.$$

Now consider the increasing sequence $\langle y_i\rangle_{i\in\mathbb{N}} = \langle 0,3,60,1197,\ldots\rangle$ of all solutions to the equation

$$11\,y^2+1 \quad = \quad \square\ .$$

Then (denoting by $\langle x_i\rangle_{i\in\mathbb{N}}$ the associated seq. $\langle 1,10,199,3970,\ldots\rangle$):

- if $y_{2^\ell\,(2\,h+1)}$ is representable, so are $\overbrace{3\,x_h+11\,y_h \text{ and } x_h+3\,y_h}^{\textit{coprime}}$
- if y_n is representable for some $n>0$ *not* a power of 2, then the system

$$\begin{cases} X^2-11\,Y^2 &= \ 1\,, \\ 3\,X+11\,Y &= \ \upsilon^2+\upsilon\,u+3\,u^2\,, \\ X+\ 3\,Y &= \ r^2+r\,s+3\,s^2 \end{cases}$$

has a solution in $\mathbb{N}$ for which $Y\neq 0$; $\therefore$, the eq.

$$11\ \left(r^2+r\,s+3\,s^2\right)^2-\left(\upsilon^2+\upsilon\,u+3\,u^2\right)^2 \quad = \quad 2$$

has a solution such that

$$\left(r^2+r\,s+3\,s^2\right)\ \left(\upsilon^2+\upsilon\,u+3\,u^2\right) \mid y_n\ .$$

Is
$\left\{\langle u, y_{2^{2\ell+1}}\rangle \ : \ell \geqslant 2 \;\&\; u \geqslant 2^{2\ell+2}\right\}$
a Diophantine set?

An exponential-growth relation

Consider the relation

$$OD(a,\ b) \quad \overset{}{\Longleftrightarrow}_{\mathsf{Def}} \quad (\exists\, x)\big[\,(2\,x+1)\,a = b\,\big]\ .$$

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Then:

- $OD(\,u\,,\ y_{2^{2\,\ell+1}}\,) \quad \Longleftrightarrow \quad u \geqslant 2^{2\,\ell+2}\ \&\ u \mid y_{2^{2\,\ell+1}}$,
- the relation

  $$\mathcal{J}(\,u\,,\ w\,) \quad \Longleftrightarrow_{\mathrm{Def}} \quad w \in \{y_{2^{2\,\ell+1}} : \ell \geqslant 2\}\ \&\ OD(\,u\,,\ w\,)$$

  has exponential growth.

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  has exponential growth.

Is $w \in \{y_{2^{2\,\ell+1}}\ :\ \ell \geqslant 2\}$— $\boxed{\text{and, consequently, } \mathcal{J}}$ —Diophantine?

An unproven assertion implying that $w \in \{y_{2^{2\,\ell+1}} : \ell \geqslant 0\}$ is Diophantine

The following are necessary and sufficient conditions in order that

$$w \in \{y_{2^{2\,\ell+1}} \,:\, \ell \geqslant 0\}$$

holds:

An unproven assertion implying that $w \in \{y_{2^{2\ell+1}} : \ell \geqslant 0\}$ is Diophantine

The following are necessary and sufficient conditions in order that

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holds:

(I) $w > 3$

(II) $11\, w^2 + 1 = \square$

(III) $(\exists\, v\,,\ u)\ w = \ v^2 \pm v\, u + 3\, u^2$

(IV) $(\boldsymbol{r}^2 + \boldsymbol{r}\,\boldsymbol{s} + 3\,\boldsymbol{s}^2)\,(\boldsymbol{v}^2 + \boldsymbol{v}\boldsymbol{u} + 3\,\boldsymbol{u}^2) \nmid w$
for any non-trivial *integer* solution to
$$11\ \left(r^2 + r\, s + 3\, s^2\right)^2 - \left(\upsilon^2 + \upsilon\, u + 3\, u^2\right)^2 \ \ = \ \ 2$$

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☺ ☺ ☺

This results in a Diophantine specification $\boxed{\text{if}}$ "*the number of solutions to this novel quaternary quartic is finite*" !

An unproven assertion implying that $w \in \{y_{2^{2\ell+1}} : \ell \geqslant 0\}$ is Diophantine

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:-) :-) :-)

The only potential source of multiple-solutions here is condition (III), which, anyhow, is finite-fold

The issue as whether

$$11\ \left(r^2 \pm r\,s + 3\,s^2\right)^2 - \left(v^2 \pm v\,u + 3\,u^2\right)^2 \;=\; 2$$

has only finitely many solutions over $\mathbb{N}$ can be recast as the analogous problem concerning the system

$$\begin{cases} 11\,\xi^2 - \eta^2 & = \; 2 \\ \xi\,\eta & = \; \nu^2 + \nu\,t + 3\,t^2 \end{cases}$$

over $\mathbb{Z}$.

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The existence of finite-fold Diophantine representations for all listable sets thus reduces to the finitude of the set of integer points lying on a specific surface.

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